# SUBCANONICAL COORDINATE RINGS ARE GORENSTEIN

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*To our teacher Evgeny Solomonovich Golod, with gratitude*

*In all examples we consider,* [the coordinate ring of $X$] *is a Gorenstein ring; this property is one of the most powerful general tools we have in studying $X$ and its deformations.It seems to us that this point is not adequatly appreciated.*

A. Corti, M. Reid, Weighted Grassmanians.

## 1. Introduction

Let $i : X \hookrightarrow \mathbb{P} = \mathbb{P}(V)$ be a smooth connected projective variety embedded into a projective space (we are working over a fixed ground field $k$). Set $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}}(1)$ and consider the coordinate algebra

$$A = \bigoplus_{n=0}^{\infty} H^0(X, \mathcal{O}_X(n)).$$

By construction $A$ is identified with a quotient algebra $A = S/I$ where $S = Sym(V^*) = k[x_0, \ldots, x_{n-1}]$. The *Koszul homology* algebra is defined as

$$H(A) = \bigoplus_{p=0}^{n} Tor_p^S(A, k).$$

This is a (bi)graded commutative $k$-algebra, finite dimensional as a $k$-vector space.

In an inspiring paper [GKR] Gorodentsev, Khoroshkin and Rudakov prove (among others) the following elegant result. Denote by $K_X$ the canonical class of $X$.

1.1. **Theorem** ((see [GKR], Sect. 2))**.** *Suppose that*

(a) *there exists a natural $N$ such that $K_X = \mathcal{O}_X(-N)$;*
(b) $H^i(X, \mathcal{O}_X(n)) = 0$ *for all $n \in \mathbb{Z}$ and $0 < i < d := \dim X$.*

*Then $H(A)$ is Frobenius.*

Here *Frobenius* means that there exists a nondegenerate bilinear pairing $\langle\ ,\ \rangle :\ H(A) \times H(A) \longrightarrow k$, suitably compatible with the gradings, such that $\langle ab, c\rangle = \langle a, bc\rangle$.

The proof in *op. cit.* is very nice; it uses the "sphericity" of certain spectral sequence.

In this note we would like to look at this result from a slightly different perspective. Our point of departure is a fundamental result by Avramov and Golod, [AG]:

1.2. **Theorem.** *$H(A)$ is Frobenius if and only if $A$ is Gorenstein.*

In fact, Avramov and Golod work in the local situation; the passage to our graded context presents no difficulties. Indeed, according to *op. cit.*, $H(A)$ is Frobenius iff the localisation of $A$ at 0 is Gorenstein; however, $A$ is smooth outside this ideal, so this is equivalent to $A$ being Gorenstein.

So our question reduces to the Gorenstein property of $A$.

Let us say, following [GKR], that $X \subset \mathbb{P}$ is *subcanonical* if the condition (a) of Theorem 1.1 is satisfied. In the present note we prove the following

1.3. **Theorem.** *Assume* $\mathrm{char}(k) = 0$. *If $X \subset \mathbb{P}$ is subcanonical then $A$ is Gorenstein and has rational singularities.*

We establish this using certain *Key Lemma* from [H] (see Proposition 2.1) giving a sufficient condition for a singularity being Gorenstein and rational. The proof of this lemma uses Grauert-Riemenschneider theorem, and hence the characteristic zero assumption. (On the contrary, although Gorodentsev et al. assume $k = \mathbb{C}$, their proof of 1.1 works over an arbirtary field).

1.4. **Corollary.** *If $X \subset \mathbb{P}$ is subcanonical then $H(A)$ is Frobenius.*

So, the condition (b) of Theorem 1.1 is superfluous if $\mathrm{char}\, k = 0$.

The main objects of study in *op cit.* are *highest weight orbits* of a semisimple algebraic group $G$. For such $X$ the authors of [GKR] prove that (b) follows from (a).

In this case we prove that subcanonicity is *equivalent* to the Gorenstein property of $A$:

1.5. **Theorem.** *Let $X \subset \mathbb{P}(V)$ be the projectivisation of the highest weight orbit in an irreducible finite dimensional representation $V$ of a semisimple group $G$. This embedding is subcanonical if and only if the corresponding coordinate ring $A$ is Gorenstein (so, iff $H(A)$ is Frobenius).*

1.6. **Acknowledgement.** This note was written during a visit of the first author to the *Institut de Mathématiques de Toulouse*. He thanks this Institute for the hospitality.

## 2. Proof of Theorem 1.3

We keep the notation of the Introduction. The affine variety $Z := \mathrm{Spec}(A)$ is the cone over $X$; therefore it is nonsingular outside 0. It has a very nice desingularization $Y$ which is the total space of the vector bundle $\mathbb{E} = \mathcal{O}_X(-1)$. Let

$$p : Y = \mathrm{Spec}(Sym_{\mathcal{O}_X}(\mathbb{E}^*)) \longrightarrow X \tag{1}$$

be the projection.

The embedding $\mathcal{O}_{\mathbb{P}(V)}(1) \longrightarrow V$ defines an embedding $Y \longrightarrow X \times V$; the projection to the second factor has image $Z \subset \mathrm{Spec}(Sym\ V^*) = V$ and the map

$$\pi : Y \longrightarrow Z \tag{2}$$

is a desingularization.

Recall the following

2.1. **Proposition** (see [H]). *Let $\pi : Y \longrightarrow Z$ be a proper birational map with $Y$ smooth and $Z$ normal. Let $\omega_Y$ be the sheaf of higher differentials on $Y$. Assume there exists a morphism $\phi : \mathcal{O}_Y \to \omega_Y$ such that $\pi_*\phi : \pi_*\mathcal{O}_Y \to \pi_*\omega_Y$ is an isomorphism. Then $Z$ is Gorenstein and has rational singularities.*

We wish to apply this to our desingularization $\pi : Y \to Z$. Note that $Z = \mathrm{Spec}(A)$ is normal.

The short exact sequence of vector bundles on $Y$

$$0 \longrightarrow p^*\mathbb{E} \longrightarrow T_Y \longrightarrow p^*T_X \longrightarrow 0 \tag{3}$$

yields an isomorphism

$$\omega_Y = p^*(\omega_X \otimes \mathbb{E}^*). \tag{4}$$

We wish to calculate the global sections of $\omega_Y$. First of all, we have

$$p_*\omega_Y = p_*p^*(\omega_X \otimes \mathbb{E}^*) = \omega_X \otimes \mathbb{E}^* \otimes Sym_{\mathcal{O}_X}\mathbb{E}^* = \bigoplus_{n\geq 1} \omega_X \otimes \mathcal{O}_X(n) \tag{5}$$

since $p$ is an affine morphism.

2.2. **Proof of Theorem 1.3.** Let $\omega_X = \mathcal{O}_X(-N)$. One has an obvious map

$$\mathcal{O}_X = \omega_X \otimes \mathcal{O}_X(N) \hookrightarrow \bigoplus_{n\geq 1} \omega_X \otimes \mathcal{O}_X(n) = p_*\omega_Y$$

which gives by adjunction a map $\phi : \mathcal{O}_Y \longrightarrow \omega_Y$.

We will check now that $\phi$ induces an isomorphism of the global sections. Applying to $\phi$ the direct image functor $p_*$ we get a morphism

$$p_*(\phi) : \bigoplus_{n\geq 0} \mathcal{O}_X(n) \longrightarrow \bigoplus_{n\geq 1} \omega_X \otimes \mathcal{O}_X(n) \tag{6}$$

which is obviously a map of $p_*(\mathcal{O}_Y)$-modules. By definition it carries $1 \in p_*(\mathcal{O}_Y)$ to a generator of $\omega_X(N) = \mathcal{O}_X$, so the map $p_*(\phi)$ carries isomorphically the summand $\mathcal{O}_X(n)$ of the left-hand side to the summand $\omega_X \otimes \mathcal{O}_X(N+n)$ of the right-hand side. For $n < N$ one has on the right-hand side

$$\Gamma(X, \omega_X \otimes \mathcal{O}_X(n)) = \Gamma(X, \mathcal{O}_X(n-N)) = 0,$$

so $p_*(\phi)$ induces an isomorphism of the global sections.

## 3. Homogeneous case

Let now $G$ be a semisimple Lie group, $V$ a simple finite dimensional highest weight $G$-module, $v \in V$ be a highest weight vector. Let $P$ be the stabilizer of $\mathbb{C}v$ in $\mathbb{P}(V)$. This is a parabolic subgroup of $G$. A $G$-equivariant embedding $i : X := G/P \longrightarrow \mathbb{P}(V)$ is induced.

The closure $Z$ of $Gv$ is a cone in $V$. We have $Z = \operatorname{Spec}(A)$ where $A$ is the homogeneous coordinate ring of $X = G/P$ with respect to $i$.

In this case the converse of the theorem 1.3 is valid. One has

3.1. **Theorem.** *The space $Z$ is Gorenstein iff $\omega_X = \mathcal{O}_X(-N)$ for some $N$.*

Note that the conclusion of the Theorem is not true for an arbitrary (nonhomogeneous) $X$ (for example it follows easily from the results of Mukai [M] that a generic curve of genus 7 embedded canonically in $\mathbb{P}^6$ has a Gorenstein coordinate ring).

*Proof.* The dualizing complex of $Z$ can be calculated as

$$\omega_Z = R\operatorname{Hom}_{SV^*}(A, SV^*)[\dim V - \dim Z] \tag{7}$$

(the shift is chosen so that $\omega_Z$ is concentrated in degree 0 when $A$ is Cohen-Macaulay).

Its cohomology keeps the grading of $SV^*$ and $A$; therefore, if $A$ is Gorenstein so that $\omega_Z$ is an invertible $A$-module, it has to be isomorphic to $A$.

Choose an isomorphism $\theta : A \longrightarrow \omega_Z$.

We now apply the Duality isomorphism, see [Ha], VII.3.4, to the proper morphism $\pi : Y \to Z$. It gives, in particular, an isomorphism

$$\operatorname{Hom}_{D(Y)}(F, \pi^! G)) \xrightarrow{\sim} \operatorname{Hom}_{D(Z)}(R\pi_* F, G) \tag{8}$$

for any $F \in D^-_{qc}(Y),\ G \in D^+_c(Z)$.

We apply this to $F = \mathcal{O}_Y$ and $G = \omega_Z$. By a general result of Kempf [K] $Z$ has rational singularities, so $R\Gamma(Y, \mathcal{O}_Y) = \Gamma(Y, \mathcal{O}_Y) = A$. Moreover, $\pi^!(\omega_Z) = \omega_Y$. Thus, Duality isomorphism gives us

$$\operatorname{Hom}_{D(Y)}(\mathcal{O}_Y, \omega_Y)) \xrightarrow{\sim} \operatorname{Hom}_{D(Z)}(\mathcal{O}_Z, \omega_Z). \tag{9}$$

We see that the map $\theta : A \to \omega_Z$ is adjoint to a map $\theta_Y : \mathcal{O}_Y \to \omega_Y$ which in turn can be rewritten as a morphism

$$\theta_X : \mathcal{O}_X \to p_*(\omega_Y) = \bigoplus_{n\geq 1} \omega_X(n). \tag{10}$$

We intend to prove now that each direct component $\theta_{X,n} : \mathcal{O}_X \to \omega_X(n)$ is either isomorphism or vanishes. This will immediately imply the theorem.

Note that the formula (7) shows that the group $G$ naturally acts on $\omega_Z$. We claim that $\theta : A \to \omega_Z$ is necessarily $G$-equivariant.

In fact, the $G$-action on $A$-module $\omega_Z$ is compatible with $G$-action on $A$:

$$g(ax) = g(a)g(x),\ g \in G,\ a \in A,\ x \in \omega_Z.$$

Another $G$-module structure on $\omega_Z$ compatible with the $G$-action on $A$ is given by $\theta$. These two actions define two group homomorphisms

$$\rho_1, \rho_2 : G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(\omega_Z).$$

The “difference” between the two defined by the formula

$$\rho_{12} : g \mapsto \rho_1(g^{-1}) \circ \rho_2(g)$$

gives rise to a crossed homomorphism $\rho_{12} : G \to \mathrm{Aut}_A(\omega_Z) = \mathbb{C}^*$. Since the action of $G$ on $\mathbb{C}^*$ is trivial and $G$ is semisimple, $\rho_{12}$ is trivial, which means that $\theta$ is $G$-equivariant.

Let us show that the maps $\theta_Y$ and $\theta_X$ obtained from $\theta$ via Duality isomorphism, are also $G$-equivariant.

Choose $g \in G$ and let $g_X : X \to X,\ g_Y : Y \to Y,\ g_Z : Z \to Z$ denote the corresponding automorphisms of the varieties.

An action of $g \in G$ on $\mathcal{O}_Z$ and $\omega_Z$ are expressed as isomorphisms $g_Z^*(\mathcal{O}_Z) \to \mathcal{O}_Z$ and $g_Z^*(\omega_Z) \to \omega_Z$. Since $\theta$ is equivariant, it gives rise to a commutative diagram

$$\begin{tikzcd}
g_Z^*(\mathcal{O}_Z) \arrow[r, "g_Z^*\theta"] \arrow[d] & g_Z^*(\omega_Z) \arrow[d] \\
\mathcal{O}_Z \arrow[r, "\theta"] & \omega_Z
\end{tikzcd} \tag{11}$$

The map $\theta_Y$ can be described as the composition

$$\mathcal{O}_Y \longrightarrow \pi^! R\pi_*(\mathcal{O}_Y) = \pi^!\mathcal{O}_Z \longrightarrow \pi^!\omega_Z,$$

so that it suffuces to check that the first morphism is $G$-equivariant. The latter can be expressed as the commutativity of the diagram

$$\begin{tikzcd} g_Y^*(\mathcal{O}_Y) \arrow[r] \arrow[d] & g_Y^*(\pi^! R\pi_*(\mathcal{O}_Y)) \arrow[d] \\ \mathcal{O}_Y \arrow[r] & \pi^! R\pi_*(\mathcal{O}_Y) \end{tikzcd} \tag{12}$$

for each $g \in G$, and this follows from the relations

$$g_Y^* \pi^! = \pi^! g_Z^*, \quad g_Z^* R\pi_* = R\pi_* g_Y^*.$$

All this proves that $\theta_Y$ is $G$-equivariant; the similar fact for $\theta_X$ is even more transparent.

We have already understood that the components $\theta_{X,n}$ of the map $\theta_X : \mathcal{O}_X \longrightarrow \bigoplus \omega_X(n)$ are $G$-equivariant. This implies that the map of fibers at $1P \in G/P$ is $P$-equivariant. The fibers are one-dimensional representations of $P$; any $P$-morphism is either zero or an isomorphism. This proves the theorem. $\square$

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